The Stability of Fluid Motion.

By T. H. HAVELOCK, F.R.S.

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1. The following notes on the stability of fluid motion arose from a desire to use the energy method, introduced by Reynolds and modified by Orr, as a measure of the comparative degree of stability of various types of flow under different boundary conditions. A few examples are worked out to illustrate this point of view: in § 5 a case which resembles the flow of a stream with a free surface; in § 7 flow which approximates to a uniform stream between fixed walls without slipping at the walls; in §§ 6, 8 motion with other boundary conditions. Before proceeding to these, it seems desirable to give a short account of the method in the form in which it is used later, together with some remarks on its relation to the classical method of small vibrations.

2. We shall consider only two-dimensional motion of an incompressible viscous fluid limited by the planes $y = \pm a$. Let the steady state under an assigned forcive and given boundary conditions be specified by a velocity, U, parallel to the axis of x. Let the disturbed state have velocity components $(\mathrm{U}+u, v)$ and let the additional pressure be p. Then, by taking the difference of the two sets of hydrodynamical equations for the two states and neglecting squares and products of the additional velocities, we have

$$\frac{\partial u}{\partial t}+\mathrm{U}\frac{\partial u}{\partial x}+v\frac{\partial \mathrm{U}}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\nabla^2 u,$$

$$\frac{\partial v}{\partial t}+\mathrm{U}\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\nabla^2 v, \qquad (1)$$

together with the equation of continuity.

It is convenient to introduce non-dimensional variables given by

$$x = a\xi; \qquad y = a\eta; \qquad a\tau = \bar{\mathrm{U}}t;$$

where $\bar{\mathrm{U}}$ is the mean velocity over the cross-section in the steady state. Further, we write $\bar{\mathrm{U}}\mathrm{U}$ instead of U, and take the current function of the additional velocity to be $\bar{\mathrm{U}}a\psi$. Eliminating p from the two equations (1), we obtain

$$\mathrm{R}\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\xi}\right)\nabla^2\psi-\mathrm{R}\mathrm{U}''\frac{\partial\psi}{\partial\xi} = 2\nabla^4\psi, \qquad (2)$$

where U″ is written for $d^2\mathrm{U}/d\eta^2$, and R is Reynolds' number $2a\bar{\mathrm{U}}/\nu$. There are in addition the appropriate boundary conditions for the disturbing function, ψ. The classical method of examining the stability of a given distribution U consists in assuming a solution of (2) of the form $\exp.\{i(n\tau+p\xi)\}f_n(\eta)$. For any arbitrary real value of p, the corresponding possible forms of $f_n(\eta)$ and values of n are found from (2) together with the boundary conditions. The distribution U may be said to be thoroughly stable if every possible value of n has a positive imaginary part, and if this holds for all positive values of R.

The usual boundary conditions, which we shall assume in the first place, are $u = 0$, $v = 0$, or

$$\psi = 0\,; \qquad \partial\psi/\partial\eta = 0: \qquad \eta = \pm 1. \tag{3}$$

From the work of Kelvin, Rayleigh, Orr, Hopf, and others, it may be taken that the simple shearing motion, $U = 1+\eta$, is thoroughly stable in this sense; and probably a similar conclusion holds for motion under a constant force or pressure gradient, namely $U = \frac{3}{2}(1-\eta^2)$.

There are various possible explanations of the well-known divergence between these results and the behaviour of actual fluids. In the first place, it is obvious that the physical properties, whether of the fluid or of the walls, are inadequately specified in the mathematical statement of the problem. But, apart from this, the disturbances have been supposed small, and second order terms neglected. Again, in a system of this type, a disturbance may be small initially and may converge ultimately to zero, but may be very large at intermediate times, and may thus give rise to practical instability.

The energy method of Reynolds is in a different category from these in that it takes the mathematical problem as it stands and does not necessarily involve the actual magnitude of the disturbance; in fact, it forms a new criterion or measure of degree of stability. The energy of the disturbance being defined by

$$E = \tfrac{1}{2}\rho a^2 \bar{U}^2 \iint \left\{ \left(\frac{\partial\psi}{\partial\xi}\right)^2 + \left(\frac{\partial\psi}{\partial\eta}\right)^2 \right\} d\xi\, d\eta, \tag{4}$$

we have from (2) and (3), after integrating by parts,

$$\frac{dE}{dt} = \tfrac{1}{2}\mu \bar{U}^2 \left[R \iint U' \frac{\partial\psi}{\partial\xi}\frac{\partial\psi}{\partial\eta}\, d\xi\, d\eta - 2 \iint (\nabla^2\psi)^2\, d\xi\, d\eta \right]. \tag{5}$$

Here dE/dt means the rate of increase of E in a region whose end boundaries move with the steady velocity U. We may replace this by $\partial E/\partial t$ for a region with fixed ends, and we shall then have additional terms on the right of (5) denoting flux of energy across these ends. The latter terms may be omitted under conditions which cover the usual cases: namely, either the disturbance is periodic in ξ, or it is limited or localised so that ψ and its derivatives converge sufficiently rapidly to zero for $\xi = \pm\infty$. We shall assume such conditions to hold in what follows, and references to boundary conditions mean those which hold at the planes $\eta = \pm 1$.

Reynolds' method of using (5) to determine a criterion of stability consisted in assuming a suitable form for ψ and finding the least value of R for which the right-hand side of (5) is zero. It is usually stated that this method assumes turbulent motion to be already in existence, and it then gives a criterion to show whether the turbulence is increasing or decreasing momentarily; but this is somewhat misleading without defining what is meant by turbulent motion. Equation (5), as stated above, applies to any small arbitrary disturbance, neglecting terms of the second order, as in the ordinary method of small vibrations; further, U is a laminar fluid motion satisfying the usual hydrodynamical equations under the given conditions.

On the other hand, Reynolds defined U as the mean velocity at each point, taken over a small region or during a short time, and this principal or mean motion need not satisfy the ordinary equations. The extra velocities u and v then play a double part, in that they specify the disturbance, and at the same time give a measure of the turbulence; they must satisfy certain conditions as to their mean values, and then equation (5) holds in the same form when mean values are used. However, in applying it to find the criterion for flow under a constant pressure gradient, Reynolds, and Sharpe following him, did, in fact, take U to be the usual form, $C(a^2-y^2)$, for steady laminar flow. But in turbulent flow, although the variations of velocity at any point are small, yet they may cause the gradient of the mean velocity to differ appreciably from its value in laminar flow, as is obvious from a comparison of the curves of distribution of velocity across a pipe in regular and in turbulent flow.

However, it is unnecessary to dwell on this distinction, as it has been pointed out clearly by Lorentz* and other writers; further, we shall consider here only small disturbances.

3. Under these circumstances, the energy method has been given a precise and definite meaning by Orr† from the following considerations:—

If the right-hand side of (5) is positive, the energy of the disturbance is momentarily increasing. But, for a given velocity distribution, U, it may be impossible to find any function, ψ, satisfying the boundary conditions, such that that expression is positive, unless R exceeds a certain value. If such be the case, this least value of R is a critical value of definite significance. The corresponding critical disturbance is found by taking the variation of the equation

$$R\iint U'\frac{\partial\psi}{\partial\xi}\frac{\partial\psi}{\partial\eta}\,d\xi\,d\eta-2\iint(\nabla^2\psi)^2\,d\xi\,d\eta = 0, \tag{6}$$

subject to $\delta R = 0$.

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* H. A. Lorentz, 'Abhandlungen über Theor. Phys.,' vol. 1, p. 43.

† W. McF. Orr, 'Proc. Roy. Irish Acad.,' vol. 27, p. 9 (1907).

Carrying out the variation, and using the boundary conditions (3), we obtain

$$4\nabla^4\psi+2\mathrm{RU}'\frac{\partial^2\psi}{\partial\xi\,\partial\eta}+\mathrm{RU}''\frac{\partial\psi}{\partial\xi}=0. \tag{7}$$

To find the critical value of R, we assume first that ξ occurs in ψ as a factor exp. $ip\xi$, and then solve (7); using the boundary conditions, we have an equation from which we can find the least value of R for a given value of p, and finally we take the minimum value of R with respect to p.

The process has been expressed in a different form by Hamel.* Using the corresponding Green's function for the equation $\nabla^4\psi=0$, the equation (7) may be replaced by a linear integral equation for ψ, of which the required value of R is the lowest characteristic number.

Returning to equation (5), if $d\mathrm{E}/dt$ is positive for any assigned initial disturbance, it does not follow that the motion is unstable in the ordinary sense. But, if there exists an absolute minimum for R in the manner explained above, it follows that, when R is less than this value, $d\mathrm{E}/dt$ is negative for every initial disturbance, and must always remain negative. Thus the system has at least a much higher degree of stability for such values of R compared with those greater than the critical minimum. Obviously, this method does not produce any new information which is not implicit in the ordinary equations, such as equations (2) and (3); but it presents part of that information in a different form, so that the critical minimum of R may be used as a measure of the degree of stability of various distributions of velocity under different boundary conditions.

4. It is convenient to classify the boundary conditions under which the energy equation (5) is valid. For this purpose we use an alternative form derived directly from equations (1), with the ordinary notation

$$p_{xx}=-p+2\mu\frac{\partial u}{\partial x};\quad p_{xy}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right);\quad p_{yy}=-p+2\mu\frac{\partial v}{\partial y}. \tag{8}$$

We have

$$\frac{d\mathrm{E}}{dt}=\int\{u\,(lp_{xx}+mp_{xy})+v\,(lp_{xy}+mp_{yy})\}\,ds-\rho\iint uv\frac{d\mathrm{U}}{dy}\,dx\,dy$$
$$+\iint\left\{p_{xx}\frac{\partial u}{\partial x}+p_{yy}\frac{\partial v}{\partial y}+p_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}dx\,dy, \tag{9}$$

where ds is a line element of the boundary and (l, m) the normal.

We have specified the conditions at the end boundaries, and we are concerned now with the planes $y=\pm a$. It follows that we get the energy equation (5), without any surface integrals expressing transfer of energy across the boundaries, with the following combinations:

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* G. Hamel, 'Gött. Nachr., Math. Phys. Klasse,' 1911, p. 261.

(i) $u = 0$, $v = 0$; (ii) $u = 0$, $p_{yy} = 0$; (iii) $v = 0$, $p_{xy} = 0$; (iv) $p_{xy} = 0$, $p_{yy} = 0$.

We may also verify that, under these conditions, the variation of (6) leads to the same differential equation (7).

5. Most of the fluid motions whose stability has been examined, come under case (i) of the above. A different case of special interest is a stream with a free upper surface, the conditions at the upper surface being as in (iv). These conditions, however, do not lead to simple expressions in terms of the disturbing function, ψ; moreover it is not permissible to regard the upper free surface as rigorously plane. We therefore, following Kelvin,* replace the problem by one which is very nearly the same but is more easily specified; it may be described as a broad river flowing over a perfectly smooth inclined plane bed, the upper surface being fitted by a parallel plane cover moving with the water in contact with it. The conditions at the upper surface then come under case (iii) of the previous section.

We take the origin in the upper surface in this case, so that a is the depth of the stream and R is $a\mathrm{U}/\nu$. The steady state is given by

$$\mathrm{U} = \tfrac{3}{2}(1-\eta^2). \tag{10}$$

Using this in (7) and assuming ψ to be proportional to $e^{ip\xi}$, the differential equation becomes

$$\left(\frac{d^2}{d\alpha^2}-1\right)^2 \psi - k\left(2\alpha\frac{d\psi}{d\alpha}+\psi\right) = 0, \tag{11}$$

where $\alpha = p\eta$, and $k = 3i\mathrm{R}/2p^3$.

The boundary conditions are $u = 0$, $v = 0$ at the bed of the stream, and $v = 0$, $p_{xy} = 0$ at the upper surface; these reduce to

$$\begin{aligned} &\psi = 0,\ d^2\psi/d\alpha^2 = 0; \qquad \alpha = 0; \\ &\psi = 0,\ d\psi/d\alpha = 0; \qquad \alpha = p. \end{aligned} \tag{12}$$

Equation (11) was solved by Orr for flow between two fixed planes with u and v zero at both boundaries, and it was found necessary to consider only solutions in even powers of α. We shall require here the corresponding solutions in odd powers. Writing a solution in the form

$$\psi = \Sigma\, \mathrm{A}_n\alpha^n/n!$$

we have the sequence relation

$$\mathrm{A}_{n+4} - 2\mathrm{A}_{n+2} + \{1-(2n+1)k\}\mathrm{A}_n = 0. \tag{13}$$

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* Kelvin, 'Math. and Phys. Papers,' vol. 4, p. 330.

Denoting by ψ_0, ψ_1, ψ_2, ψ_3 the solutions beginning with 1, α, α^2, α^3 respectively, it follows from (12) that the boundary conditions lead to

$$\psi_1 d\psi_3/d\alpha - \psi_3 d\psi_1/d\alpha = 0 \tag{14}$$

where α has to be replaced by p.

Calculating the coefficients far enough to give sufficient accuracy for our purpose, we have

$$\begin{aligned}\psi_1 = {} & \alpha + 2\alpha^3/3\,! + (3+3k)\,\alpha^5/5\,! + (4+20k)\,\alpha^7/7\,! \\ & + (5+70k+33k^2)\,\alpha^9/9\,! + (6+180k+366k^2)\,\alpha^{11}/11\,! \\ & + (7+385k+2029k^2+627k^3)\,\alpha^{13}/13\,! + (8+728k+7832k^2+9672k^3)\,\alpha^{15}/15\,! \\ & + (9+1260k+24030k^2+73500k^3+16929k^4)\,\alpha^{17}/17\,! + \ldots,\end{aligned}$$

$$\begin{aligned}\psi_3 = {} & \alpha^3/3\,! + 2\alpha^5/5\,! + (3+7k)\,\alpha^7/7\,! + (4+36k)\,\alpha^9/9\,! \\ & + (5+110k+105k^2)\,\alpha^{11}/11\,! + (6+260k+894k^2)\,\alpha^{13}/13\,! \\ & + (7+525k+4213k^2+2415k^3)\,\alpha^{15}/15\,! + (8+952k+14552k^2 \\ & \qquad + 28968k^3)\,\alpha^{17}/17\,! + \ldots\end{aligned}$$

Forming equation (14) we have

$$\begin{aligned} & 2/3\,! + 8p^2/5\,! + 32p^4/7\,! + 128p^6/9\,! + (512+192k^2)\,p^8/11\,! \\ & + (2048+2244k^2)\,p^{10}/13\,! + (8192+19456k^2)\,p^{12}/15\,! \\ & + (32768+139264k^2)\,p^{14}/17\,! + (131072+901120k^2 \\ & \qquad + 129024k^4)\,p^{16}/19\,! + \ldots = 0. \end{aligned} \tag{15}$$

Only even powers of k appear in this equation, thus giving a check upon the arithmetic; further, the terms independent of k may be summed. Taking the least root of (15) as an equation for k^2, we have approximately

$$\mathrm{R}^2 = \frac{\sinh 2p - 2p}{9p^5\left(\dfrac{192}{11\,!} + \dfrac{2244p^2}{13\,!} + \dfrac{19456p^4}{15\,!} + \dfrac{139264p^6}{17\,!} + \dfrac{901120p^8}{19\,!} + \ldots\right)}. \tag{16}$$

Instead of forming an equation for the minimum value of R, it is simpler to find it by trial. We find, with sufficient accuracy, that it occurs near $p^2 = 11$, and then, approximately,

$$\mathrm{R} = 96. \tag{17}$$

The corresponding value, found by Orr, for flow under similar conditions but with a fixed plane at the upper surface, is 117. We conclude then that flow in an open canal has a lower degree of stability than flow between fixed planes.

Turning to experimental results, the number usually quoted for flow through a tube is 2000 approximately. This was obtained chiefly from experiments with smooth glass tubes; a much lower number, of the order of 400, has been found from metal tubes. The only available direct results for flow in an open stream appear to be those given by Hopf,[*] who found R to be of the order of 300. These results agree in character with the theoretical calculations, which is all that could be expected.

It is of interest to note that this appears to contradict a statement by Reynolds[†] in one of his earlier papers. He classes separately circumstances conducive to steady motion and those conducive to unsteady motion: among the former a free surface, and in the latter solid bounding walls. However, this opinion seems to be based on visual observation of eddies caused by the wind beneath the oiled surface of water. "At a sufficient distance from the windward edge of an oil-calmed surface there are always eddies beneath the surface, even when the wind is light. . . . Without oil I was unable to perceive any indication of eddies."

This introduces a different property of a boundary surface, namely, that of initiating disturbances. The mathematical statement ignores this property and specifies only control of the velocity functions: the disturbances are supposed to be initiated by some extraneous agency, and it is tacitly assumed that all types of disturbance are equally probable. It may be, for instance, that the theoretical results for flow through pipes should be compared with experiments on rough pipes rather than those with perfectly smooth walls. However, we may conclude that a solid boundary is conducive to stability in so far as it ensures that there is no slipping of the fluid in contact with it.

6. In determining the minimum value of R from the differential equation (7), there are only two factors: the distribution of steady velocity, U, and the boundary conditions for the disturbance. The comparison in the previous section, between an open stream and flow between fixed walls, involved changes in both these factors. We may separate the effect of the boundary conditions by assuming the same value of U as in (10), but expressing the property of the supposed moving plane in contact with the upper surface by $u = 0$, $v = 0$, instead of by $v = 0$, $p_{xy} = 0$. To anticipate the argument of the next sections, we should expect a value of R intermediate between 96 and 117.

We have the same equation (11) for ψ, together with $\psi = 0$, $d\psi/d\alpha = 0$ at $\alpha = 0$, and $\alpha = p$. It follows that only the solutions ψ_2 and ψ_3 are involved, and we have

$$\psi_2\, d\psi_3/d\alpha - \psi_3\, d\psi_2/d\alpha = 0, \tag{18}$$

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[*] L. Hopf, 'Ann. der Phys.,' vol. 32, p. 777 (1910).

[†] O. Reynolds, 'Scientific Papers,' vol. 2, pp. 157, 59. See also A. H. Gibson, 'Phil. Mag.,' vol. 25, p. 81 (1913).

when $\alpha = p$. The series for ψ_3 is given in § 5; also we have

$$\begin{aligned}\psi_2 = {} & \alpha^2/2!+2\alpha^4/4!+(3+5k)\,\alpha^6/6!+(4+28k)\,\alpha^8/8! \\ & +(5+90k+65k^2)\,\alpha^{10}/10!+(6+220k+606k^2)\,\alpha^{12}/12! \\ & +(7+455k+3037k^2+1365k^3)\,\alpha^{14}/14!+(8+840k+10968k^2 \\ & \qquad +17880k^3)\,\alpha^{16}/16!+\ldots.\end{aligned}$$

The boundary equation (18) leads to

$$\begin{aligned} & 2/4!+8p^2/6!+32p^4/8!+128p^6/10!+(512+280k^2)\,p^8/12! \\ & \quad +(2048+3136k^2)\,p^{10}/14!+(8192+25216k^2)\,p^{12}/16! \\ & \qquad +(32768+174080k^2)\,p^{14}/18!+\ldots = 0. \qquad (19)\end{aligned}$$

The minimum value of R seems to occur for about $p^2 = 12$, though it is not a sharply defined minimum; however, with a similar approximation as in previous cases, we find the critical minimum of R to be 110.

7. It is well known that, when fluid motion through a tube has changed from laminar to turbulent flow, the distribution of mean velocity over the cross-section alters so that the velocity becomes more nearly uniform over the greater part of the section, while falling to zero at the walls. This suggests a study of the comparative stability when the distribution of velocity alters in this manner, the boundary conditions being unchanged.

However, it must be noted that we assume the distribution to be a steady state which has been acquired under a law of force, which may be determined from the hydrodynamical equations, so as to give the required form for U.

A simple form, which illustrates the points in question, is

$$U = (1+1/2n)(1-\eta^{2n}). \qquad (20)$$

As n is made larger, the velocity approximates more closely to the mean velocity, $\bar{U}$, over the greater part of the cross-section, while remaining zero at the walls. The corresponding law of force is, in the usual notation,

$$X = \nu(4n^2-1)(U/a^2)\,\eta^{2n-2}. \qquad (21)$$

The greater the value of n, the more is the field of force concentrated near the walls, quite apart from the value of the viscosity. The flow approximates to a uniform stream, but retaining the condition of zero velocity at the walls.

The usual case of flow under a uniform field of force is given by $n = 1$. It is sufficient for comparison to work out another numerical case, say $n = 2$. We have then

$$U = \tfrac{5}{4}(1-\eta^4). \qquad (22)$$

Equation (7) becomes

$$\left(\frac{d^2}{d\alpha^2}-1\right)^2\psi-2k\left(2\alpha^3\frac{d\psi}{d\alpha}+3\alpha^2\psi\right)=0, \tag{23}$$

where $k = 5i\mathrm{R}/8p^5$.

The boundary conditions are

$$\psi=0\,;\qquad d\psi/d\alpha=0\,;\qquad \alpha=\pm p.$$

Solving (23) by a power series $\Sigma \mathrm{A}_n\alpha^n/n\,!$, we have

$$\mathrm{A}_{n+6}=2\mathrm{A}_{n+4}-\mathrm{A}_{n+2}+2k(n+1)(n+2)(2n+3)\,\mathrm{A}_n. \tag{24}$$

As in the simpler cases, it is sufficient to choose fundamental solutions involving only even powers of α; denoting these by ψ_0 and ψ_2 we have

$$\begin{aligned}\psi_0 = {} & 1+\alpha^2/2\,!+\alpha^4/4\,!+(1+12k)\,\alpha^6/6\,!+(1+192k)\,\alpha^8/8\,!\\ & +(1+1032k)\,\alpha^{10}/10\,!+(1+3552k+20160k^2)\,\alpha^{12}/12\,!+(1+9492k\\ & +696960k^2)\,\alpha^{14}/14\,!+(1+21504k+8162256k^2)\,\alpha^{16}/16\,!+\ldots\end{aligned}$$

$$\begin{aligned}\psi_2 = {} & \alpha^2/2\,!+2\alpha^4/4\,!+3\alpha^6/6\,!+(4+168k)\,\alpha^8/8\,!+(5+1656k)\,\alpha^{10}/10\,!\\ & +(6+8184k)\,\alpha^{12}/12\,!+(7+28392k+574560k^2)\,\alpha^{14}/14\,!\\ & +(8+78960k+11204352k^2)\,\alpha^{16}/16\,!+(9+188496k\\ & +102266496k^2)\,\alpha^{18}/18\,!+\ldots.\end{aligned}$$

From the boundary condition

$$\psi_0\,d\psi_2/d\alpha-\psi_2\,d\psi_0/d\alpha=0,$$

we obtain the equation

$$\begin{aligned}& p+2p^3/3\,!+8p^5/5\,!+32p^7/7\,!+128p^9/9\,!+512p^{11}/11\,!\\ & +(2048+129024k^2)\,p^{13}/13\,!+(8192+3280896k^2)\,p^{15}/15\,!\\ & +(32768+7753296k^2)\,p^{17}/17\,!+\ldots=0.\end{aligned} \tag{25}$$

Using this as an equation for R, we find by trial that the minimum value occurs near $p^2 = 3$; and the critical minimum value of R is 280 approximately.

The corresponding value for the ordinary parabolic distribution ($n = 1$) is 117. Thus, the critical value of R increases as the flow approximates more closely to a uniform stream, without slipping at the walls; and, in this sense, the motion becomes increasingly stable.

8. It has been stated that, under the boundary conditions $u = 0$, $v = 0$, there is thorough stability, in the ordinary sense, for simple shearing motion and probably also for laminar flow between fixed planes. In view of the behaviour of actual fluids in similar conditions, another suggestion has been put forward by Hopf.* He proposes to express the influence of a wall by making the extra normal pressure, due to the disturbance, constant at the

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* L. Hopf, 'Ann. der Phys.,' vol. 59, p. 538 (1919).

wall, together with no tangential slipping; in fact, his boundary conditions come under case (ii) of § 4, namely $u = 0$, $p_{yy} = 0$. With these assumptions, he applies the method of small vibrations to simple shearing motion between a fixed plane and a parallel moving plane. It appears that the motion is unstable for disturbances whose wave-length exceeds a certain value; for smaller wave-lengths it is stable or unstable according to the value of R. Thus the motion is not thoroughly stable. Without discussing how far these assumptions express the behaviour of actual fluids and boundaries, we may see how they affect the energy method.

We shall take the case of laminar flow between fixed planes, for which the previous calculations are available.

The stream function ψ satisfies equation (11), and the boundary conditions are

$$u = 0; \qquad -p+2\mu\,\partial v/\partial y = 0.$$

From the equations (1), these are equivalent to

$$u = 0; \qquad \rho v\, d\mathrm{U}/dy-\mu\,\partial^2 u/\partial y^2 = 0,$$

or, in the present notation,

$$d\psi/d\alpha = 0; \qquad d^3\psi/d\alpha^3-2k\alpha\psi = 0; \qquad \alpha = \pm p. \tag{26}$$

Using the solutions ψ_0 and ψ_2, these give

$$\psi_0'(\psi_2'''-2kp\psi_2)-\psi_2'(\psi_0'''-2kp\psi_0) = 0, \tag{27}$$

where accents denote differentiation with respect to α.

From the previous work, this equation involves odd powers of k. But k is $3i\mathrm{R}/4p^3$ and we have to determine R in terms of p from (27). It follows that in this case there is no real solution of the problem of finding the critical minimum of R.

It seems probable that it is only those motions which are completely stable in the ordinary theory which lead also to a real minimum for R. The suggestion may be stated in this manner: if a fluid motion is thoroughly stable when considered by the method of small vibrations applied to equation (2) and the boundary conditions, then it also possesses a real minimum value of R found from equation (7) and the boundary conditions. It has been pointed out that the latter equation is derived directly from the former, and it may be presumed that the minimum value of R depends in some manner upon the rates of decay of elementary vibrations and so may be used as a measure of the degree of stability of the system.